THE TWENTY-THIRD SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-third summer meeting and eighth colloquium of the Society were held at Harvard University during the week September 4–8, 1916. Monday and Tuesday were devoted to the summer meeting proper, two sessions being held on each day for the presentation and discussion of papers. The colloquium opened on Wednesday morning and extended to Friday afternoon. An account of the colloquium appears in this number of the BULLETIN.

The following ninety-nine members attended the summer meeting:

Professor L. D. Ames, Professor R. C. Archibald, Professor G. N. Armstrong, Professor C. S. Atchison, Professor Clara L. Bacon, Professor A. A. Bennett, Professor G. D. Birkhoff, Professor Henry Blumberg, Professor Maxime Bôcher, Professor C. L. Bouton, Professor E. W. Brown, Dr. T. H. Brown, Dr. R. W. Burgess, Professor W. D. Cairns, Professor B. H. Camp, Dr. A. B. Chace, Professor C. W. Cobb, Professor F. N. Cole, Professor L. L. Conant, Professor J. L. Coolidge, Dr. A. R. Crathorne, Professor Louise D. Cummings, Professor C. H. Currier, Professor D. R. Curtiss, Professor E. W. Davis, Dr. C. E. Dimick, Dr. C. R. Dines, Professor Arnold Dresden, Professor Otto Dunkel, Professor W. P. Durfee, Professor W. C. Eells, Professor John Eiesland, Professor L. P. Eisenhart, Professor G. C. Evans, Mr. G. W. Evans, Professor H. B. Fine, Professor T. S. Fiske, Dr. L. R. Ford, Dr. M. G. Gaba, Professor W. C. Graustein, Dr. G. M. Green, Professor M. W. Haskell, Professor C. N. Haskins, Dr. Olive C. Hazlett, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor L. M. Hoskins, Professor E. V. Huntington, Professor W. A. Hurwitz, Professor Dunham Jackson, Professor W. W. Johnson, Dr. E. A. T. Kircher, Professor A. E. Landry, Mr. B. B. Libby, Dr. Joseph Lipka, Professor H. P. Manning, Professor J. L. Markley, Professor Helen A. Merrill, Dr. A. L. Miller, Professor H. B. Mitchell, Professor U. G. Mitchell, Professor C. L. E. Moore, Professor C. N. Moore, Professor G. D. Olds, Dr. G. A. Pfeiffer, Professor J. M. Poor, Professor L. H. Rice, Professor R. G. D. Richardson, Professor E. D. Roe, Jr., Dr. A. R. Schweitzer, Dr. Caroline E.

Seely, Dr. H. M. Sheffer, Dr. L. L. Silverman, Professor H. E. Slaught, Professor Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Professor Sarah E. Smith, Professor Virgil Snyder, Professor R. P. Stephens, Professor R. B. Stone, Professor E. B. Stouffer, Professor H. W. Tyler, Professor A. L. Underhill, Professor E. B. Van Vleck, Professor Oswald Veblen, Professor Roxana H. Vivian, Professor C. A. Waldo, Professor A. G. Webster, Dr. Mary E. Wells, Dr. C. E. Wilder, Professor F. B. Williams, Professor F. N. Willson, Dr. Euphemia R. Worthington, Mr. W. C. Wright, Professor C. H. Yeaton, Professor J. W. A. Young, Professor J. W. Young, Professor Alexander Ziwet.

The President of the Society, Professor E. W. Brown, occupied the chair, being relieved by Vice-Presidents E. R. Hedrick and Virgil Snyder. The Council announced the election of the following persons to membership in the Society: Mr. Herman Betz, Cornell University; Mr. J. A. Bigbee, High School, Little Rock, Ark.; Mr. Hillel Halperin, Vanderbilt University; Dr. J. R. Kline, University of Pennsylvania; Professor J. J. Luck, University of Virginia; Dr. F. J. McMackin, Dartmouth College. Seven applications for membership in the Society were received.

Through the generosity of Harvard University the Freshman Dormitories and dining room were thrown open for the use of the Society during the meeting. On Monday noon the members of the Society were shown the collection of mathematical models belonging to the University. On Wednesday afternoon a visit was paid to the University Library, and on Wednesday evening to the Observatory. Resolutions were adopted at the meeting expressing the thanks of the Society for the hospitality of the University and its officers.

Fraternal greetings were exchanged by cable with the Scandinavian mathematicians assembled at Stockholm. A vote of congratulation was tendered to the Secretary on his twenty-first year of service in that capacity.

The twenty-fifth anniversary of the broadening out of the Society into a national organization and the founding of the BULLETIN were celebrated at the banquet on Monday evening, at which eighty-four members and friends were present. Brief addresses were made by Professors Fiske, Johnson, Fine, Birkhoff, Hedrick, Webster, Coolidge, and the Secretary.

On Tuesday evening Professor D. E. Smith entertained the Society with an instructive account of "The relation of the history of economics to the history of arithmetic problems."

The following papers were read at this meeting:

(1) Professor J. C. FIELDS: "Direct derivation of the complementary theorem."

(2) Dr. C. A. FISCHER: "Note on the order of continuity of functions of lines."

(3) Dr. OLIVE C. HAZLETT: "On the theory of associative division algebras."

(4) Professor W. C. EELLS: "A statistical study of eminent mathematicians."

(5) Professor J. L. COOLIDGE: "The characteristic numbers of real algebraic plane curves."

(6) Dr. R. W. BURGESS: "The comparison of a certain case of the elastic curve with its approximation."

(7) Professor G. A. MILLER: "Orders of operators of congruence groups modulo $2^r 3^s$."

(8) Professor JOHN EIESLAND: "Sphere geometry (third paper)."

(9) Dr. A. J. KEMPNER: "Generalization of a theorem on transcendental numbers."

(10) Professor C. N. MOORE: "On the developments in Bessel's functions."

(11) Professor ARNOLD DRESDEN: "Supplementary note on the second derivatives of an extremal integral."

(12) Professor L. E. DICKSON: "An extension of the theory of numbers by means of correspondences between fields."

(13) Professor A. G. WEBSTER: "On a theory of acoustic horns."

(14) Professor E. H. MOORE: "On properly positive Hermitian matrices."

(15) Professor L. P. EISENHART: "Deformations of transformations of Ribaucour."

(16) Professors F. R. SHARPE and VIRGIL SNYDER: "On (2, 2) point correspondence between two planes."

(17) Professor F. H. SAFFORD: "Surfaces of revolution in the theory of Lamé's products."

(18) Professor DUNHAM JACKSON: "Note on the parametric representation of an arbitrary continuous curve."

(19) Professor DUNHAM JACKSON: "Note on representations of the partial sum of a Fourier series."

(20) Professor L. H. RICE: "Determinants of many dimensions."

(21) Dr. L. R. FORD: "Regular continued fractions."

(22) Professor M. W. HASKELL: "The eliminant of a system of forms."

(23) Professor E. V. HUNTINGTON: "A simple substitute for Duhamel's theorem."

(24) Professor E. V. HUNTINGTON and Dr. J. R. KLINE: "Sets of independent postulates for betweenness."

(25) Professor G. D. BIRKHOFF: "Dynamical systems with two degrees of freedom (second paper)."

(26) Professor E. B. VAN VLECK: "Non-loxodromic substitutions in n variables."

(27) Dr. L. I. HEWES: "Nomograms of adjustment."

(28) Mr. H. C. M. MORSE: "A theorem on the linear dependence of analytic functions of a single variable."

(29) Dr. G. A. PFEIFFER: "Note on the linear dependence of analytic functions."

(30) Dr. G. M. GREEN: "On the linear dependence of functions of one variable."

(31) Professor C. L. BOUTON: "Iteration and group theory."

(32) Dr. G. M. GREEN: "On the general theory of surfaces."

(33) Dr. W. V. GARRETSON: "On the asymptotic solution of the non-homogeneous linear differential equation of the nth order. A particular solution."

(34) Dr. CAROLINE E. SEELY: "On series of biorthogonal functions."

(35) Professor A. B. FRIZELL: "Lemma for a new method of generating alephs."

(36) Professor JOHN EIESLAND: "Transformation theory of the flat complex and its associated line complex."

(37) Dr. A. R. SCHWEITZER: "On a type of quasi-transitive functional equations (second paper)."

(38) Dr. A. R. SCHWEITZER: "A problem in quasi-transitive functional equations."

(39) Dr. A. R. SCHWEITZER: "Some theorems on quasi-transitive functional equations."

(40) Dr. A. R. SCHWEITZER: "On the analogy between functional equations and geometric order relations."

(41) Dr. T. H. GRONWALL: "On the power series for $\log (1 + z)$."

(42) Dr. T. H. GRONWALL: "A problem in geometry connected with the analytic continuation of a power series."

(43) Dr. T. H. GRONWALL: "On the convergence of Binet's factorial series for log $\Gamma(z)$ and $\psi(z)$."

(44) Dr. T. H. GRONWALL: "On the zeros of the function $\beta(z)$ associated with the gamma function."

Mr. Morse was introduced by Professor Osgood. The papers of Professor Fields, Dr. Fischer, Professor Miller, Dr. Kempner, Professor Dickson, Professor E. H. Moore, Professor Safford, Dr. Hewes, Dr. Garretson, Dr. Seely, Professor Frizell, Dr. Schweitzer, and Dr. Gronwall were read by title. Abstracts of the papers appear below; the abstracts are numbered to correspond to the titles in the list above.

1. There is nothing in the nature of the complementary theorem which should require for its proof a preliminary determination of a formula for the actual number of the conditions imposed on a rational function of (z, u) of given form by a given set of orders of coincidence corresponding to a specific value of z. The proof here given by Professor Fields dispenses with the aid of such a formula and thus shortens by one step the approach to the theorem. The general complementary theorem then presents itself at a very early stage in the development of the theory of the algebraic functions, the only steps preliminary to its proof, besides those implied in the proof of the existence theorem for the branches of an algebraic function, being those involved in the determination of the reduced form for a rational function of (z, u), those involved in the proof of the integral character relative to the element $z - a$ of the principal coefficient of a rational function of (z, u) which is adjoint for the value $z = a$, and those involved in the proof of what we may call the theorem of principal residues which is derived therefrom and which is contained in the statement: The necessary and sufficient conditions in order that a rational function $H(z, u)$ may have orders of coincidence which are complementary adjoint to a given set of orders of coincidence for the value $z = a$ are obtained on equating to 0 the principal residue relative to $z = a$ in the product of $H(z, u)$ by the general rational function conditioned by the given set of orders of coincidence.

2. Dr. Fischer's paper appears in full in the present number of the BULLETIN.

3. There is a famous theorem to the effect that the only linear associative algebras over the field of real numbers in which division is uniquely possible are the field of real numbers, the field of ordinary complex numbers, and real quaternions, the most recent of the numerous proofs of this being one by Professor Dickson. In his paper the theorem is a special case of a theorem for a certain class of algebras (called Type A) over a general algebraic field F which has a subalgebra S which can be exhibited as an algebra L over a field K with the r^2 units $i^s j^k$ ($s, k = 1, \cdots, r$), where

$$ji = \theta(i)j, \quad j^r = g(i), \tag{1}$$

where i is an element of S satisfying in K a uniserial abelian equation of degree r, with the roots $\theta(i), \cdots, \theta^r(i) = i$.

Dr. Hazlett's paper considers linear associative division algebras over a general algebraic field F, and finds that a necessary and sufficient condition that such an algebra be of Type A is that θ in (1) be the root of a certain algebraic equation, which we will call the Θ-equation. Then, from some properties of the Θ-equation, this paper proves that, if a linear associative division algebra over an algebraic field be of rank n, it is of order $m \cdot n$ where $m \leqq n$. Of this theorem, Frobenius's theorem is a corollary. It also follows that, if a linear associative division algebra, of rank n, over an algebraic field F, contain a number i which satisfies a uniserial abelian equation of degree n, then any number in the algebra is a polynomial in j with coefficients in $F(i)$, where (1) holds. There are numerous other theorems, some of which are generalizations of well known facts about quaternions.

4. From a representative group of standard histories of mathematics and of general encyclopedias, Professor Eells by the space method selects a group of one hundred most eminent mathematicians of history, discusses the reliability of the selection, and classifies and discusses the men selected according to nationality, period of history in which they flourished, principal branches of mathematics to which they contributed, and other features.

5. A real algebraic plane curve has two sorts of characteristic numbers. First, there are the total or Plücker characteristics, which are invariant for the general plane collineation. Sec-

ond, there are the real characteristics, as the number of real inflections, real double tangents, etc. These are invariant for real collineations. The only known identical equation connecting these latter is the equation of Klein. In Professor Coolidge's paper it is shown that, taking total and real characteristics together, the only universally valid equations are the equations of Plücker and Klein, and others deducible from them.

6. In the usual discussion of the elastic curve in elementary text-books of physics and engineering, a certain term in the differential equation is neglected and the curve obtained as the solution of this approximate equation is said to be a good substitute for the real elastic curve. In the case of a bow or a vertical column this approximate solution is a cosine curve, which is satisfactory if the given constants are the force and the deflection. In the more common problem, however, a bow or column of known length is bent by a known force. Dr. Burgess shows that if a formula for the deflection in terms of the force and the length is deduced from the cosine curve to apply to this case, the result is only about one half the true deflection as found from the real elastic curve.

7. The $\phi(m)$ positive integers which do not exceed the natural number m and are prime to m, when these numbers are combined by multiplication modulo m, constitute a group which is simply isomorphic with the group of isomorphisms of the cyclic group of order m. The principal object of Professor Miller's paper is to show how the order of each of the operators of this group can be directly obtained from the form of the corresponding number by means of general theorems relating to the automorphisms of an abelian group, whenever m is of the form 2^r3^s. Attention is called to the fact that in H. Weber's Lehrbuch der Algebra, second edition, volume 2 (1899), page 65, the exponent to which 5 belongs modulo 2^s is found by a very special method, and that a somewhat similar method is followed by P. Bachmann in the first part of his Niedere Zahlentheorie, 1902, page 331, while the method by means of the theorems noted above exhibits at once the exponent to which each number belongs.

8. Professor Eiesland's memoir is a continuation of a paper on the same subject read before the Society at the

meeting of December 28–29, 1915. An attempt has been made to build up a theory of flat complexes with a view to its application to sphere geometry. It has been shown that with the linear flat complex is associated a line complex which, in some important respects, is analogous to the null-system in ordinary space. The lines of this complex satisfy the system of Pfaffian equations

$$(1)\qquad \begin{matrix} Qdx_i - Pdy_i + R_i dz = 0, \\ R_{\frac{1}{2}(n-2)}dy_i - R_i dy_{\frac{1}{2}(n-2)} = 0 \end{matrix} \qquad [i = 1, 2, \cdots, \tfrac{1}{2}(n-2)],$$

where

$$P = A_{\frac{1}{2}n}z + \Sigma A_k x_k - B_0, \quad Q = A_0 z + \Sigma A_k y_k + B_{\frac{1}{2}n},$$
$$R_i = A_{\frac{1}{2}n}y_i - A_0 x_i - B_i,$$

the coefficients A_i, B_i being the constants in the equation of the flat complex

$$(2)\qquad \Sigma A_i \rho_i + \Sigma B_i \sigma_i = 0.$$

All the lines of the line complex which pass through a point form a flat pencil, i. e., they lie in a 2-flat. The relation between point and 2-flat is involutory. Every line is the axis of $\infty^{\frac{1}{2}(n-4)}$ flats of the complex (2).

9. The generalized theorem of Dr. Kempner's paper is as follows:

I. An infinite series

$$\sum_{n=1}^{n=\infty} \frac{\gamma_n}{a_n}$$

represents a transcendental number, provided

(1) $\gamma_n = p_n/q_n$ is a real rational number; p_n and q_n are either finite or grow infinite with n in some particular manner; it is for example amply sufficient if $|p_n|$, $|q_n|$ are both $< R^n$, R any fixed positive number;

(2) each denominator a_n is a positive integer and a factor of the next denominator a_{n+1};

$$(3)\qquad \lim_{n=\infty} \frac{a_{n+1}}{a_n^2} \geqq 1$$

if the limit exists; if it does not exist, then

$$\lim_{n=\infty} \inf \frac{a_{n+1}}{a_n^2} \geqq 1.$$

In 1915 (see BULLETIN, March, 1916, page 285) the author presented the theorem

I'. $$\sum_{n=1}^{n=\infty} \frac{\gamma_n}{a^{c^n}} \cdot x^n$$

represents a transcendental number, provided

(1') γ_n restricted as above under (1), but

(2') only a finite number of the $\gamma_n = 0$,

(3') a and c any integers $\geqq 2$,

(4') x any rational number $\neq 0$.

At the April, 1916, meeting at Chicago, Professor H. Blumberg announced that he had succeeded, by a modification of the author's method of proof, in entirely removing the condition (2'). Neither Professor Blumberg's proof nor the author's proof of I' makes essential use of the assumption that the denominators be of the form a^{c^n}, and analysis of the two proofs shows that it is sufficient to assume that the denominators satisfy (2) and (3) of the generalized theorem.

In Theorem I, nothing is gained by considering the series

$$\sum_{n=1}^{n=\infty} \frac{\gamma_n}{a_n} \cdot x^n$$

for rational x, since this is included in I. Theorem I could not be proved before Professor Blumberg's welcome contribution had eliminated (2').

10. Professor Moore's paper appeared in full in the October BULLETIN.

11. In a paper presented to the Society at its meeting in Chicago, December, 1914, Professor Dresden gave formulas for the second derivatives of the extremal integrals arising in the theories of the integrals

$$\int f(x; y_1, \cdots, y_n; y_1', \cdots, y_n')dx$$

and

$$\int f(y_1, \cdots, y_n; y_1', \cdots, y_n')dt.$$

A certain lack of symmetry in the formulas for the second of these theories made them rather unsuitable for applications and undesirable from a theoretical point of view. By using the "normal" solutions of the system of Jacobi differential

equations recently introduced by Professor Bliss, it has been found possible to remove this lack of symmetry and to simplify the formulas very materially. It is the object of the present note to derive these simplified formulas.

12. Professor Dickson's paper will appear in full in an early number of the BULLETIN.

13. Since no boundary problem for an unclosed non-plane surface has been solved with the exception of Kelvin's spherical bowl, there is as yet no theory of horns of finite length. As such a theory is eminently desirable not only for musical instruments but for reception apparatus, including the phonograph, megaphone, and a great variety of acoustical instruments, Professor Webster has been driven to a very coarse approximate theory, which works surprisingly well in practice. Two assumptions are made, first that the cross-section of the horn is infinitesimal with respect to the wave length, and varies as any given function of the distance from a certain section; second that at the open end Helmholtz's assumption for tubes of infinitesimal cross-section may be used. The first assumption leads to a linear differential equation, and when the section varies as a power of the axial distance this may be integrated by Bessel functions of fractional index. The cases of cylinders, cones, and hyperbolic horns such as are used in the orchestra come under this case. The assumption for the end, even when the diameter is half the wave length instead of an infinitesimal part of it, is found to agree within a few per cent with experimental determinations. Any horn is characterized by four constants, and may be replaced by any other horn whatever having the same constants.

14. The new theory of linear integral equations in general analysis, of which a special instance is Hilbert's theory of limited symmetric bilinear forms in infinitely many variables, under development by Professor Moore, is based on a properly positive Hermitian matrix or numerically valued function ϵ or $\epsilon(s, t)$ of two variables s, t ranging independently over a general class $\mathbf{P}$, viz., a function ϵ of such a nature that (1) $\overline{\epsilon(s, t)} = \epsilon(t, s)$ for every s, t of $\mathbf{P}$; (2) every finite principal minor is positive, i. e., for every set $(p_1, p_2, \cdots, p_n)$ of a

finite number n of distinct elements of **P**, the determinant of the n^2 functional values $\epsilon(p_i, p_j)(i, j = 1, 2, \cdots, n)$ is positive. For the Hilbert instance, the class **P** is the class of positive integers, and the function ϵ is δ, $\delta(s, t) = d_{st} = 1(s = t)$, $0(s \neq t)$.

The present paper of Professor Moore presents various absolute and relative theorems of existence of such matrices ϵ. As an example, if **P** is the class of all real valued functions p or $p(x)$ on the interval 01, ϵ or $\epsilon(s, t) = \exp \int_0^1 s(x)t(x)dx$ is such a matrix ϵ. This example is of especial importance in that for the corresponding instance of the general theory we meet for the first time a non-trivial instance of an integration process whose integrand functions are themselves functional operations; this "integration over a function space" is secured in virtue of the fact that the integration process involves in its definition no assumed metrical features of the range of integration.

15. When a system of spheres involves two parameters, their envelope consists, in general, of two sheets, say Σ and Σ_1, and the centers of the spheres lie upon a surface S. A correspondence between Σ and Σ_1 is established by making correspond the points of contact on the same sphere. When the lines of curvature on Σ and Σ_1 correspond, Σ_1 is said to be in the relation of a transformation of Ribaucour with Σ, and vice versa. It is a known property of envelopes of spheres that if the surface of centers S be deformed and the spheres be carried along in the deformation, the points of contact of the spheres with their envelope in the new position are the same as before deformation. Professor Eisenhart determines the transformations of Ribaucour which in deformations of the surface of centers S become transformations of Ribaucour. The surfaces Σ in this case have the same spherical representation of their lines of curvature as isothermic surfaces. The paper will be published in the October number of the *Transactions*.

16. Marletta (*Palermo Rendiconti*, volume 17 (1903)) has discussed the (2, 2) correspondence between the points of two planes in which a line is transformed into a conic or a cubic. In the paper of Professors Sharpe and Snyder the two images

of a point in one plane are the intersections of a corresponding line and conic in the other. The image of a line is a quintic of genus 2. The correspondences are classified according to the number of basis points the nets of conics and the quadratic systems of lines may have.

The results obtained by Marletta and those known earlier all appear as particular cases.

17. In the investigation of surfaces of revolution in the theory of Lamé's products it is necessary to find a function of two variables such that a certain known function of the original one shall be the sum of two functions, one of the first variable and the other of the second. After the original function has been determined, the resolution of the known function into a sum as stated has required separate treatment for each case considered. In this paper Professor Safford has obtained the general expression for the component parts of the sum in terms of the original function. The meridian curves are in general Weierstrass $\wp$-function curves. The paper will be sent to the *Archiv der Mathematik und Physik*, in which several related papers have already appeared.

18. A continuous curve may be defined by a pair of equations

$$x = f(t), \qquad y = \varphi(t),$$

where the functions f and φ are defined and continuous throughout some interval, and are not both constant throughout that interval. For some discussions it is convenient to suppose further that they are not simultaneously constant throughout any sub-interval of their interval of definition. The question arises, whether this assumption implies a restriction on the curve itself, that is, on the set of points (x, y) given by the equations, or whether it is a restriction merely on the particular parametric representation employed. It is obvious that the latter is the case, if the number of intervals of constancy is finite.

It was proved by Fréchet (*Rendiconti Palermo*, volume 22) that this remains true, even when there are infinitely many of the intervals in question. Professor Jackson gives a different proof of the same theorem.

19. In the general theory of the convergence of expansions in series associated with ordinary linear differential equations,

as developed by Birkhoff (*Transactions*, 1908) and others, the sum of the first n terms of the series is represented by a contour integral in the plane of the complex parameter. The contour integral, in the fundamental particular case of Fourier's series, must of course be identical in value with the ordinary real formula for the partial sum of the series. In his second note, Professor Jackson enumerates the steps in the process of verifying this identity by direct evaluation of the contour integral, without going back to the individual terms which make up the sum. The Fourier's series can be derived either from a system of the first order or from one of the second order, and the work is carried through for both cases.

20. An extended definition of a determinant is given by Professor Rice, which applies to determinants of more than three dimensions and enables us to remove the restriction in Cayley's law of multiplication and to set up a new case in Scott's law of multiplication. New formulas are obtained for the known process of decomposition of a determinant into determinants of fewer dimensions, and a new process called crossed decomposition is described. Fresh light is thrown upon the function known as a "determinant-permanent," a limitation hitherto thought necessary being done away. Finally a generalization to p dimensions is made of Metzler's theorem in two dimensions concerning a determinant each of whose elements is the product of k factors.

21. A continued fraction

$$b_0 \pm \frac{a_1}{b_1 \pm} \frac{a_2}{b_2 \pm} \cdots \frac{a_n}{b_n \pm} \cdots$$

is called regular if $a_i = \pm 1$ or $\pm \sqrt{-1}$, and

$$b_i = m_i + n_i \sqrt{-1},$$

where m_i and n_i are real integers. In Dr. Ford's paper the classic pentahedral division of half-space corresponding to the group of Picard is employed to give a geometrical interpretation of the convergents of the continued fraction.

It is shown how any such continued fraction can be determined by means of a curve lying in the half-space. Explicit

formulas are given for setting up the continued fraction corresponding to a given curve. By a study of the generating curve general theorems are established relative to (1) the value of the convergents as approximations to the sum of the continued fraction, and (2) the periodicity of the continued fraction.

22. Professor Haskell gives a method of constructing the eliminant of a system of forms which leads to a direct proof of the well-known theorem that the number of solutions of a system of simultaneous equations is equal to the product of their degrees. Beginning with a system of linear forms, he uses the theory of symmetric functions to construct a series of eliminants, in which the linear forms are, one after the other, replaced by forms of higher degree, and thus arrives at the contravariant which factors into the linear factors corresponding to the various solutions of the system of forms of any degree, and without any extraneous factor.

23. Professor Huntington's note contains the following substitute for Duhamel's theorem, which may seem simpler than the substitutes already published in the *Annals of Mathematics* by Osgood, R. L. Moore, and Bliss: Suppose that a required quantity P is associated with a real interval, $x = a$ to $x = b$, in a way that suggests dividing the interval into n small parts, or elements, and regarding P as the sum of n separate contributions, one from each element. Suppose also that a set of one or more functions, $f(x)$, $F(x)$, $\cdots$, can be found such that no matter what the value of n, the contribution of a typical element, from $x = x$ to $x = x + \Delta x$, can be expressed "approximately" in the form $[f(x)\cdot F(x)\cdots]\Delta x$—that is, so that the exact value of the contribution lies between $[f_1\cdot F_1\cdots]\Delta x$ and $[f_2\cdot F_2\cdots]\Delta x$, where f_1, F_1, $\cdots$ are the least, and f_2, F_2, $\cdots$ the greatest values of $f(x)$, $F(x)$, $\cdots$ in the interval in question. Then the required quantity P can be computed as a definite integral: $P = \int_a^b [f(x)\cdot F(x)\cdots]\, dx$, provided only that the functions involved are continuous from $x = a$ to $x = b$.

24. The paper of Professor Huntington and Dr. Kline starts with the following basic list of twelve postulates for be-

tweenness: (*A*) If *AXB* then *BXA*. (*B*) If *A*, *B*, *C* are distinct, then at least one of the relations *BAC*, *CAB*, *ABC*, *CBA*, *ACB*, *BCA* will be true. (*C*) *AXY* and *AYX* cannot both be true. (*D*) If *ABC* is true, then the elements *A*, *B*, *C* are distinct. Further, if *A*, *B*, *X* and *Y* are distinct, then: (1) If *XAB* and *ABY* then *XAY*; (2) If *XAB* and *AYB* then *XAY*; (3) If *XAB* and *AYB* then *XYB*; (4) If *AXB* and *AYB* then *AXY* or *AYX*; (5) If *AXB* and *AYB* then *AXY* or *YXB*; (6) If *XAB* and *YAB* then *XYB* or *YXB*; (7) If *XAB* and *YAB* then *XYA* or *YXA*; (8) If *XAB* and *YAB* then *XYA* or *YXB*. The question then is: Given any subset *S* of the twelve postulates of this basic list, and any postulate *P* of the list, not belonging to *S*; is *P* deducible from *S*? An exhaustive answer to this question is given in 71 theorems of deducibility and 68 theorems of non-deducibility, these latter being established by the aid of 44 examples of pseudo-betweenness. It is found that eleven different sets of independent postulates may be selected from the basic list, as follows: 1, 2; 1, 5; 1, 6; 1, 7; 1, 8; 2, 4; 2, 5; 3, 5; 3, 4, 6; 3, 4, 7; 3, 4, 8, with the addition of postulates *A*, *B*, *C*, *D* in each of these sets. The paper will be offered to the *Transactions*.

25. Professor Birkhoff proves two theorems concerning the invariant points of a surface of any genus p under a one-to-one transformation into itself. The second of these theorems, in which it is assumed that the transformation leaves an area integral invariant, is based upon a modification of Poincaré's last geometric theorem. The results admit of application to dynamical systems with two degrees of freedom.

26. At the last meeting of the Society in Chicago Professor Van Vleck presented a paper on the "Composition of non-loxodromic linear substitutions, $z' = (az+b)/(cz+d)$." When expressed in unimodular form

$$z_1' = az_1 + bz_2, \quad z_2' = cz_1 + dz_2$$

with unit determinant, the non-loxodromic substitution has a real characteristic equation. Professor Van Vleck defined correspondingly a non-loxodromic linear substitution in n homogeneous variables to be one which, expressed in unimodular form, has a real characteristic equation. He then

announced his first results in extension of the preceding paper, for proof of which new methods were found necessary.

27. D'Ocagne has shown that an equation of the form

$$(1) \qquad \begin{vmatrix} f_{12} & g_{12} & h_{12} \\ f_{34} & g_{34} & h_{34} \\ f_{56} & g_{56} & h_{56} \end{vmatrix} = 0,$$

where f_{ij}, g_{ij}, h_{ij} are functions of the same two variables, may be represented by a nomogram with at most three curve nets. It is proposed by Dr. Hewes to generalize the form of equation (1) by the use of the equation

$$(2) \qquad \begin{vmatrix} f_{ij} & g_{ij} & h_{ij} \\ f_{kl} & g_{kl} & h_{kl} \\ f_{mn} & g_{mn} & h_{mn} \end{vmatrix} = 0,$$

where the subscripts may take on the values 0, 1, 2, $\cdots$, 6. There result nomograms requiring rotation of the index and of wide application.

28. Mr. Morse proves the following theorem, which is due to Professor Osgood:

Let $f_1(t), f_2(t), \cdots, f_p(t)$ be p functions of the complex variable t, analytic in a region S of the t-plane. Then a necessary and sufficient condition for the linear dependence of these functions is the identical vanishing of a p-rowed determinant whose ith row ($i = 1, 2, \cdots, p$) is $f_i^{(\lambda_1)}(t_1)$, $f_i^{(\lambda_1-1)}(t_1)$, $\cdots$, $f_i'(t_1)$, $f_i(t_1)$, $f_i^{(\lambda_2)}(t_2)$, $f_i^{(\lambda_2-1)}(t_2)$, $\cdots$, $f_i'(t_2)$, $f_i(t_2)$, $\cdots$, $\cdots$, $f_i^{(\lambda_\mu)}(t_\mu)$, $f_i^{(\lambda_\mu-1)}(t_\mu)$, $\cdots$, $f_i'(t_\mu)$, $f_i(t_\mu)$, where $t_1, t_2, \cdots, t_\mu$ are independent variables and the upper indices denote differentiation.

The theorem includes the ordinary wronskian theorem for analytic functions as a special case, and the method of proof affords a new demonstration for the latter theorem.

29. Dr. Pfeiffer proves the theorem given by Mr. Morse by showing that successive differentiation of the determinant in question, and subsequent equating of all the independent variables, reduces the determinant to the wronskian of the given functions.

30. In his first paper Dr. Green establishes a condition under which the vanishing of the determinant of the two preceding papers is sufficient for the linear dependence of the functions involved, these functions not being required to be analytic. The existence of only those derivatives which appear in the said determinant is required, the only restriction being the non-vanishing of a certain first minor in the determinant. The proof is analogous to the one given by Bôcher* for the corresponding wronskian theorem.

31. Professor Bouton's note gives a discussion of a method of finding all the continuous one-parameter groups which contain a given individual point transformation in n variables. This given transformation is taken in the form $x_i' = x_i + \psi_i(x)$ where the ψ_i are convergent power series beginning with terms of the second degree. In the course of the work the result of iterating the given transformation k times is derived.

32. In extending theorems of metric differential geometry to projective, it is necessary to substitute for the congruence of normals to a surface a congruence which is likewise uniquely determined by the surface, but projectively. Wilczynski's directrix congruence of the second kind might be made to serve the purpose; but this congruence has not the peculiar projective property of the normal congruence, whose developables intersect the surface in a conjugate net. In his second paper, Dr. Green defines and characterizes geometrically a congruence which has this desired property. In a previous paper† he defined a relation R, by means of which to every line through a point P of the surface is made to correspond a unique line lying in the tangent plane, and vice versa—the correspondence depending upon the parametric net. Let the surface be referred to its asymptotic curves, and let l be a line in the tangent plane, l' the corresponding line through the point P. Project the asymptotic curves on the tangent plane from any point on l', and construct the conics C_1 and C_2 which osculate these projections at the point P. Then there exists but one pair of lines l, l' such that the intersec-

* "Certain cases in which the vanishing of the wronskian is a sufficient condition for linear dependence," *Transactions Amer. Math. Society*, vol. 2 (1901), pp. 139–149, Theorem II.

† "On rectilinear congruences and nets of curves on a surface." Abstract in the BULLETIN, vol. 22 (1916), p. 274.

tions of l with the asymptotic tangents are the double points of the involution determined by the two pairs of points in which l cuts the conics C_1 and C_2. Let L_1 and L_2 be the intersections of the asymptotic tangents with the line l thus uniquely determined, and denote by D_1 and D_2 the intersections of the asymptotic tangents with the directrix of the first kind. Calling P_1 the harmonic conjugate of D_1 with respect to P and L_1, and P_2 the harmonic conjugate of D_2 with respect to P and L_2, one thus obtains a line m joining the points P_1 and P_2 and lying in the tangent plane to the surface. Corresponding to the line m, in the relation R, there exists a line m' passing through the point P. For every point P of the surface exists such a line m', and the congruence of these lines has the property that its developables cut the surface in a conjugate net.

Among other results arrived at may be mentioned a geometric characterization of the tetrahedron of reference which gives rise to a certain canonical development of the non-homogeneous coordinates of a point of the surface. This development was given by Darboux without any geometric characterization. The points L_1 and L_2 of the preceding investigation are two of the vertices of the tetrahedron, and the line m' is an edge.

33. Dr. Garretson's paper treats of the irregular integrals of the non-homogeneous linear equation of any order, where the roots of the characteristic equation are distinct. The irregular point is taken as the point at infinity. Two theorems given by Dini for the homogeneous equation have been generalized so as to apply to the non-homogeneous equation and combined in one theorem. To this end certain ideas given by Horn have been employed. Use is also made of the solutions obtained by Professor C. E. Love for the homogeneous equation. The paper leads to the asymptotic development, in the Poincaré sense, of a particular solution of the non-homogeneous equation.

34. A necessary and sufficient condition for the essentially uniform convergence of series of orthogonal functions in two variables

$$\sum_{i=1}^{\infty} \frac{\varphi_i(x)\varphi_i(y)}{\lambda_i}$$

has been given by Lauricella. Miss Seely considers the corresponding problem for series of the form

$$\sum_{i=1}^{\infty} \frac{\varphi_i(x)\psi_i(y)}{\lambda_i},$$

where the φ's and ψ's are a biorthogonal system of functions, and obtains conditions of which Lauricella's theorem is a special case.

35. In the November BULLETIN* Professor Frizell used a simple special case of the following lemma. Given a triply infinite partition $u_{\lambda\mu\nu}$ of an infinite set of symbols v_ι selected from an ω-series a_i $(i, \iota, \lambda, \mu, \nu = 1, 2, \cdots)$ and a non-enumerable well-ordered set P of different permutations among the members of an ω-series; if now the elements $u_{\lambda\mu\nu}$ are exhibited in a rectangular array of infinite series, wherein λ, μ denote the row and column respectively:

$$\begin{array}{llllllllll}
u_{111}, & u_{112}, & \cdots; & u_{121}, & u_{122}, & \cdots; & u_{131}, & u_{132}, & \cdots; & \cdots \\
u_{211}, & u_{212}, & \cdots; & u_{221}, & u_{222}, & \cdots; & u_{231}, & u_{232}, & \cdots; & \cdots \\
u_{311}, & u_{312}, & \cdots; & u_{321}, & u_{322}, & \cdots; & u_{331}, & u_{332}, & \cdots; & \cdots \\
u_{411}, & u_{412}, & \cdots; & u_{421}, & u_{422}, & \cdots; & u_{431}, & u_{432}, & \cdots; & \cdots \\
\cdots & & & \cdots & & & \cdots & & &
\end{array}$$

then the permutations P operating on the indices ν (each pair λ, μ being held fast) produce new matrices $\mathfrak{M}$ yielding permutations $\mathfrak{P}$ which can not be put into one-to-one correspondence with the set P. The present note is devoted to the proof of the general proposition.

36. To the conformal group in S_{n-1} (n even) corresponds in $\bar{S}_{n-1}$ a group of contact transformations $G_{\frac{1}{2}(n+1)(n+2)}$ which transforms the ∞^n flats

$$x_i = ay_i + b_i, \quad z = \Sigma c_i y_i + d,$$
$$az = \Sigma c_i x_i + ad - \Sigma c_i b_i \tag{1}$$

inter se. The group also preserves asymptotic lines on a surface. An important subgroup of this group is considered by Professor Eiesland, viz.:

* Vol. 22, No. 2, p. 73.

1. The group of the complex $a + d = 0$ which carries null-flats into null-flats, with which is closely connected
2. The group of the associated line complex, and also
3. The group of parameters which shows how the flats (1) are transformed when the surface elements of $\bar{S}_{n-1}$ are subjected to the transformations of $G_{\frac{1}{2}(n+1)(n+2)}$. This group is the group of the quadric surface

$$e = ad - \Sigma b_i c_i$$

so that the geometry of flats is equivalent to a geometry on a quadric surface in a space of $n + 1$ dimensions.

The following theorem is also proved:

It is always possible by means of a linear transformation to reduce the system of Pfaffian equations

$$Qdx_i - Pdy_i + R_i dz = 0, \quad R_{\frac{1}{2}(n-2)}dy_i - R_i dy_{\frac{1}{2}(n-2)} = 0$$

to any one of the following forms:

(1) $dx_i - zdy_i + y_i dz = 0, \quad y_{\frac{1}{2}(n-2)}dy_i - y_i dy_{\frac{1}{2}(n-2)} = 0,$

or

(2) $zdx_i - x_i dz = 0, \quad x_i dy_{\frac{1}{2}(n-2)} - x_{\frac{1}{2}(n-2)}dy_i = 0,$

or

(3) $dx_i = 0.$

37. In a former paper Dr. Schweitzer considered quasi-transitive functional equations of the type

(1) $f(u_1, u_2, \cdots, u_{n+1}) = f(y_1, y_2, \cdots, y_{n+1}),$

where

$$u_i = f(t_1, t_2, \cdots, t_n, x_i), \quad y_i = \sum_{k=1}^{n+1} m_{ik}x_k,$$

and it was shown that if

$$f(x_1, x_2, \cdots, x_{n+1}) = \sum_{k=1}^{n+1} \alpha_k x_k$$

then

$$\sum_{k=1}^{n+1} \alpha_k = 0$$

and $\alpha_2, \alpha_3, \cdots, \alpha_{n+1}$ respectively satisfy n algebraic equations of the nth degree with coefficients in the field of

$$M_{st} = m_{st} - m_{1t} \quad (s, t = 2, 3, \cdots, n + 1).$$

Each of the roots α is, moreover, a rational function of α_{n+1} with regard to M_{st} $(s, t \neq 1)$.

In the present paper, the problem of deriving conditions on the M_{st} such that the n equations of the nth degree possess the same roots $\alpha_2, \alpha_3, \cdots, \alpha_{n+1}$ is discussed. In this case, the resulting single equation, if irreducible in the field of certain of the M's in which the coefficients of the equation lie, is regular with reference to this field. Conversely, if the n relations (derived from the equation (1)) in terms of $\alpha_2, \cdots, \alpha_{n+1}$ and rational with regard to M_{st}, are subjected to the substitutions of a regular group on the symbols $\alpha_2, \cdots, \alpha_{n+1}$, then from the resulting n sets of n relations it follows that $\alpha_2, \cdots, \alpha_{n+1}$ are roots of the same equation. This consideration leads to sufficient conditions for the desired properties of the $\alpha_2, \cdots, \alpha_{n+1}$. For example, when $n+1 = 3, 4, 5$ the author obtains four theorems corresponding to the four regular groups on two, three and four symbols:

I. If

$$f(u_1, u_2, u_3) = f(-bx_2, -bx_3, -ax_1 + (a-b)x_3)$$

then α_2, α_3 are roots of the quadratic $\xi^2 - a\xi + ab = 0$.

II. If

$$f(u_1, u_2, u_3, u_4) = f(y_1, y_2, y_3, y_4),$$
$$f(v_1, v_3, v_4, v_2) = f(y_1, y_3, y_4, y_2),$$

where

$$v_i = f(t_1, t_2, x_i, t_3)$$

then $\alpha_2, \alpha_3, \alpha_4$ are roots of the cubic $\xi^3 - a\xi^2 + ab\xi - ac = 0$ provided

$$\sum_{t=1}^{4} M_{st} = 0 \ (s = 2, 3, 4), \quad M_{21} = 0, \quad M_{31} = 0,$$

$$M_{22} = M_{43}, \quad M_{33} = M_{42}, \quad M_{23} = M_{32},$$

$$M_{22} + M_{33} + M_{44} = a, \quad M_{22} + M_{33} + M_{23} = b,$$

$$M_{22}M_{33} - M_{23}^2 = c, \quad M_{23}M_{44} = c - (M_{22}^2 + M_{33}^2).$$

III. If

$$f(u_1, u_2, u_3, u_4, u_5) = f(y_1, y_2, y_3, y_4, y_5),$$
$$f(v_1, v_3, v_2, v_5, v_4) = f(y_1, y_3, y_2, y_5, y_4),$$
$$f(w_1, w_4, w_5, w_2, w_3) = f(y_1, y_4, y_5, y_2, y_3),$$

where

$$v_i = f(t_1, t_2, t_3, x_i, t_4), \quad w_i = f(t_1, t_2, x_i, t_3, t_4)$$

then α_2, α_3, α_4, α_5 are roots of the biquadratic $\xi^4 - a\xi^3 + ab\xi^2 - ac\xi + ad = 0$ provided

$$\sum_{t=1}^{5} M_{st} = 0 \ (s = 2, 3, 4, 5), \quad M_{21} = 0, \quad M_{31} = 0,$$

$$M_{41} = 0, \quad M_{22} = M_{52}, \quad M_{33} = M_{53}, \quad M_{44} = M_{54},$$

$$M_{32} = M_{42}, \quad M_{23} = M_{43}, \quad M_{24} = M_{34},$$

$$M_{22} + M_{33} + M_{44} + M_{55} = a$$

and M_{55}, b, c, d are certain rational functions of M_{22}, M_{33}, M_{44}, M_{23}, M_{32}, M_{24}.

IV. This theorem is analogous to theorem III. Presumably one obtains for $n + 1 > 5$ theorems on algebraic equations corresponding to every regular substitution group on n symbols.

38. Dr. Schweitzer proposes the following problem: Let

$$f(u_1, u_2, \cdots, u_{n+1}) = \phi(x_1, x_2, \cdots, x_{n+1}), \tag{1}$$

where

$$u_i = f(t_1, t_2, \cdots, t_n, x_i) \quad (i = 1, 2, \cdots, n + 1);$$

to determine the function f under the following conditions: If

$$f(x_1, x_2, \cdots, x_{n+1}) = f_0(x_1, x_2, \cdots, x_{n+1})$$

is a solution of (1) so also is

$$f(x_1, x_2, \cdots, x_{n+1}) = f_0(x_1, \cdots, x_k, x_{i_{k+1}}, \cdots, x_{i_{k+j}}, x_{k+j+1}, \cdots, x_{n+1}),$$

where $i_{k+1}, \cdots, i_{k+j}$ each range over $k + 1, k + 2, \cdots, k + j$ and are distinct and

$$\begin{pmatrix} (k+1), & \cdots, & (k+j) \\ i_{k+1}, & \cdots, & i_{k+j} \end{pmatrix}$$

is any given substitution of any given group G on the symbols $x_{k+1}, \cdots, x_{k+j}$. Two important special cases of this problem are (I) the determination of a class of solutions which are permuted among themselves when subjected to the substitutions of the group G when the latter is supposed, e. g., regular; (II) the determination of a solution of (1) which is invariant under the substitutions of G. The linear func-

tions discussed in the preceding paper of the author yield solutions of problems of type (I), whereas corresponding to (II) the following theorem is proved: Given any substitution group G

$$G: \quad \begin{Bmatrix} X_2, & \cdots, & X_n \\ X_{i_2}, & \cdots, & X_{i_n} \end{Bmatrix} (X_{n+1})$$

there exists a group of quasi-transitive functional equations, namely,

$$E \begin{Bmatrix} 1, & 2, & 3, & \cdots, & n, & n+1 \\ 1, & i_2, & i_3, & \cdots, & i_n, & n+1 \end{Bmatrix}$$

simply isomorphic with G and which have as a common particular solution a function which is formally invariant under the substitutions of the group G.

39. Dr. Schweitzer proves the following theorems:

I. If

$$f\{z, f(x, y)\} = f\{y, f(x, z)\}, \quad \phi\{f(x, y), x\} = \psi(y),$$

$$f\{\phi(x, y), x\} = \psi(y), \quad f\{\psi(x), \psi(y)\} = f(y, x),$$

then there exists $\chi(x)$ such that

$$\chi f(x, y) = \chi(x) - \chi(y), \quad \chi\phi(x, y) = \chi(x) - \chi(y) + c,$$

$$\chi\psi(x) = c - \chi(x)$$

where c is an arbitrary constant. Therefore

$$\phi\{\phi(x, y), \phi(x, z)\} = \phi(z, y)$$

and

$$f\{f(x, y), f(x, z)\} = f(z, y).$$

II. If

$$f\{u_1, u_2, \cdots, u_{n+1}\} = f\{\theta(x_1), \cdots, \theta(x_{n+1})\}$$

where

$$u_i = f\{t_1, t_2, \cdots, t_n, x_i\} \quad (i = 1, 2, \cdots, n+1)$$

then

$$\psi f(x_1, \cdots, x_{n+1}) = \Theta\{\psi(x_1) - \psi(x_2), \cdots, \psi(x_1) - \psi(x_n)\} + k(\psi(x_i) - \psi(x_{n+1})) + c_1,$$

$$\psi\theta(x) = -k\psi(x) + c_2$$

where k, c_1, and c_2 are arbitrary constants and Θ is an "arbitrary" function of $(n-1)$ variables.

III. If

$$f\{u_1, u_2, \cdots, u_{n+1}\} = f\{\alpha_1(x_1), \alpha_2(x_3), \alpha_3(x_4), \cdots, \alpha_n(x_{n+1}), \alpha_{n+1}(x_2)\}$$

then

$$\psi f(x_1, x_2, \cdots, x_{n+1}) = \sum_{i=1}^{n+1} a_i \psi(x_i) + l,$$

$$\psi\alpha_i(x) = m_i\psi(x) + p_i \quad (i = 1, 2, \cdots, n+1),$$

where

$$\sum_{i=1}^{n+1} a_i p_i = 0, \quad \sum_{i=1}^{n+1} a_i = 0, \quad a_2 = m_{n+1}, \quad a_{n+1} = m_1,$$

$$a_{j+1} = \frac{m_2 m_3 \cdots m_j}{m_1^{j-1}} \cdot m_{n+1} \quad (j = 2, 3, \cdots, n-1)$$

and $m_1^n = m_2 m_3 \cdots m_n m_{n+1}$. The m's and p's are constants arbitrary apart from the conditions indicated.

40. The object of Dr. Schweitzer's fourth note is to point out an analogy that exists between geometric order relations in the foundations of geometry and functional equations. This analogy consists in interpreting functionally abstractions of geometric order relations; in other words, it is shown that certain geometric order relations and functional equations give rise to the same general abstract relations resembling the "associations" of Grassmann and Pietzker. The geometric order relations are based* mainly on the author's "right-handedness" relation $\alpha R_n \beta_1, \beta_2, \cdots, \beta_n$ and "interior" relation $\alpha I_n \beta_1 \beta_2 \cdots \beta_{n+1}$ and the functional equations arrived at belong to categories previously defined by the author, especially the "quasi-transitive" functional equations. The general logical theory of relations relevant to the preceding position is that of Veronese.

41. Dr. Gronwall proves by actual analytic continuation of the power series $\log(1+z) = z - z^2/2 + z^3/3 - \cdots$ the familiar facts that $\log(1+z)$ has its only finite singular point at $z = -1$, and that the continuation along a closed path winding once in the positive direction around $z = -1$ has the affect of increasing $\log(1+z)$ by $2\pi i$.

* Cf. *Amer. Jour. of Mathematics*, 1909, pp. 365–410.

42. Dr. Gronwall solves the following problem: given a power series in z defining an analytic function with the sole singularity $z = -1$ at finite distance, what is the minimum number m of intermediate points $a_1, a_2, \cdots, a_m$ needed to effect the analytic continuation of the given power series to a given point z_0? Writing $z = -1 + re^{\theta i}$, $-\pi \leqq \theta \leqq \pi$, it is found that the set of points z_0, for which no more than m intermediate points are necessary, is formed by the interior of the curve

$$r = \left(2 \cos \frac{\theta}{m+1}\right)^{m+1}, \quad (-\pi \leqq \theta \leqq \pi).$$

The totality of possible positions of the last intermediate point a_m for a given z_0 is also determined.

43. The four series in question are known to converge absolutely and uniformly for $R(z) \geqq \epsilon > 0$, but diverge for $R(z) < 0$ ($R(z)$ = real part of z, and ϵ arbitrarily small). Dr. Gronwall investigates the convergence on the boundary line $R(z) = 0$, and finds that all four series converge uniformly for $R(z) \geqq 0$, $|z| \geqq \epsilon$; two of the series are absolutely convergent in the same region, while the remaining two do not converge absolutely for any purely imaginary value of z.

44. In his Handbuch der Gammafunktion, Nielsen shows that the function

$$\beta(z) = \frac{1}{2}\left[\psi\left(\frac{z}{2}\right) + \psi\left(\frac{1-z}{2}\right)\right],$$

where $\psi(z) = d \log \Gamma(z)/dz$, has no real zeros, and raises the question whether any complex zeros exist. In the present paper, Dr. Gronwall establishes the existence of an infinity of such zeros, and gives their asymptotic expressions.

F. N. COLE,
Secretary.

THE CAMBRIDGE COLLOQUIUM.

THE eighth colloquium of the American Mathematical Society was held in connection with its twenty-third summer meeting at Harvard University, Cambridge, Massachusetts. At the April, 1915, meeting of the Council the invitation of

the Division of Mathematics of Harvard University to hold the summer meeting and colloquium at Harvard was accepted, and a committee of arrangements appointed, consisting of Professors Osgood, Bôcher, E. H. Moore, P. F. Smith, and the Secretary. The courses of lectures were announced in the preliminary circular of April, 1916, and printed syllabi were distributed at the meeting. The colloquium opened Wednesday morning, September 6, and continued until Friday afternoon; three lectures were delivered on each of the first two days, and four on Friday. A number of books of reference were placed on the reserved shelves of the university library for the use of those attending the colloquium, and ample opportunity for conferences was provided in Smith and Standish Halls throughout the week. The following sixty-nine persons were in attendance, a number considerably exceeding that of any previous colloquium.

Professor L. D. Ames, Professor R. C. Archibald, Professor Clara L. Bacon, Professor A. A. Bennett, Professor G. D. Birkhoff, Professor H. Blumberg, Professor M. Bôcher, Professor C. L. Bouton, Professor E. W. Brown, Dr. R. W. Burgess, Dr. A. B. Chase, Professor C. W. Cobb, Professor F. N. Cole, Professor J. L. Coolidge, Dr. A. R. Crathorne, Professor Louise D. Cummings, Professor C. H. Currier, Miss A. M. Curtis, Dr. C. E. Dimick, Professor A. Dresden, Professor Otto Dunkel, Professor J. Eiesland, Professor G. C. Evans, Mr. G. W. Evans, Professor H. B. Fine, Professor T. S. Fiske, Dr. L. R. Ford, Dr. M. G. Gaba, Professor W. C. Graustein, Dr. G. M. Green, Professor M. W. Haskell, Professor C. N. Haskins, Dr. Olive C. Hazlett, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Mr. M. T. Hu, Professor E. V. Huntington, Professor W. A. Hurwitz, Professor D. Jackson, Mr. R. Keffer, Dr. E. A. T. Kircher, Professor A. E. Landry, Mr. F. W. Loomis, Professor J. L. Markley, Professor Helen A. Merrill, Dr. A. L. Miller, Professor H. B. Mitchell, Professor C. N. Moore, Dr. F. D. Murnaghan, Dr. F. H. Murray, Dr. G. A. Pfeiffer, Dr. T. A. Pierce, Professor R. G. D. Richardson, Dr. A. R. Schweitzer, Dr. Caroline E. Seely, Dr. L. L. Silverman, Professor Clara E. Smith, Professor V. Snyder, Professor R. P. Stephens, Professor R. B. Stone, Professor A. L. Underhill, Professor E. B. Van Vleck, Professor Oswald Veblen, Professor A. G. Webster, Dr. Mary E. Wells, Dr. C. E. Wilder, Dr. Euphemia R. Worthington, Professor J. W. Young, Professor Alexander Ziwet.

Two courses of five lectures each were given:

I. Professor G. C. EVANS: "Topics from the theory and applications of functionals, including integral equations."

II. Professor O. VEBLEN: "Analysis situs."

Abstracts of the lectures follow below. The lectures will soon be published in full as Volume V of the Colloquium Series.

I.

LECTURE I. FUNCTIONALS OF CURVES AND SURFACES IN THREE DIMENSIONS; VARIATIONAL EQUATIONS.

1. Additive and non-additive functionals of curves in two and three dimensions; existence of a derivative; relation to functions of point sets.

2. The vector of a functional.

3. Variation of a functional:

$$\delta\Phi[c] = \int_c \Phi'[c|s]\delta n(s)ds.$$

$$\delta\Phi'[c|s] = \int_c \Phi''[c|s_1, s]\delta n(s_1)ds_1 + a_0(s)\delta n(s) + \sum_1^n a_i(s)\delta n^{(i)}(s).$$

$$\delta\Phi[c] = \int_c [V_x(x,y,z)dydz + V_y(x,y,z)dzdx + V_z(x,y,z)dxdy].$$

4. Functionals of surfaces; exceptional points and curves.

5. Variational equations (equations in functional derivatives of form $\Phi'[c|x, y] = F[c|\Phi, x, y]$); the equation for the Green's function; Hadamard's equation.

6. The condition for integrability; self-adjointness in the second variation.

7. Partial equations and the theory of characteristics.

8. Relation of variational equations to integro-differential equations.

LECTURE II. COMPLEX FUNCTIONALS IN SPACES OF THREE AND FOUR DIMENSIONS.

1. Complex additive functionals of curves; integrability; isogeneity, $\Phi_1 + i\Phi_2$ being isogenous to $F_1 + iF_2$; conjugateness in the ellipse belonging to the vectors V_1 of F_1 and V_2 of F_2; equations which vectors of Φ_1 and Φ_2 must independently satisfy.

2. Case where V_1 and V_2 are unit vectors parallel to x and y axes respectively.

3. The invariant $H[\Phi, \Phi']$; extension of Green's theorem; theorems on determinateness; special case when the triple product $[V_1 V_2 ds] = 0$ represents an integrable equation; reduction by a transformation to case where V_1 and V_2 are unit vectors parallel to x and y axes.

4. Generalization of Cauchy's theorem for the integral of a function of a complex variable; conjugateness; non-additive functionals.

5. Complex functionals of surfaces in space of four dimensions; integrability; isogeneity; integrals of analytic functions of two complex variables.

LECTURE III. IMPLICIT FUNCTIONAL EQUATIONS.

1. Volterra's theorem; calculation of the variation; reduction of the implicit functional equation to a linear integral equation.

2. Application to existence theorems.

3. Linear functional as limit of an integral; as Stieltjes integral; as Lebesgue integral.

4. Special forms of the linear integral equation with a parameter; the equations of the third and first kinds.

LECTURE IV. INTEGRO-DIFFERENTIAL EQUATIONS OF BÔCHER TYPE.

1. Hypothetical experiments as basis of physics.

2. Bôcher's treatment of Laplace's equation; generalization to curvilinear coordinates; in three dimensions; Poisson's equation; Cauchy and Weierstrass regions.

3. The equations

$$\int_c \lambda \frac{\partial u}{\partial n} ds = 0, \qquad \int_c \lambda \frac{\partial u}{\partial n} ds = \int\int f(x, y) d\sigma;$$

principal solution.

4. Extension of Green's theorem.

5. Calculation of the Green's function for elliptic and parabolic equations; the Green's function and Green's theorem for integro-differential equations of Volterra type, where the variables are not separable.

6. Generalization of hyperbolic differential equation; stair solutions; non-homogeneous equation.

LECTURE V. DIRECT GENERALIZATIONS OF THE THEORY OF INTEGRAL EQUATIONS.

1. Moore's general analysis; bases and closure properties; generalization of Fredholm theory, of Hilbert-Schmidt theory; mixed linear equations.

2. Commutativity; associative combination; permutability.

3. The complex operation; convergence properties; differential properties.

4. Volterra's theory of permutable functions; fundamental rôle of the Volterra relation; fractions of composition; logarithms of composition; symbolical treatment of integro-differential equations.

II.

LECTURE I. ONE-DIMENSIONAL ANALYSIS SITUS.

1. Analysis situs has two chief divisions, dealing (1) with continuity considerations and (2) with combinatorial ones. These lectures are to deal primarily with (2), making enough use of (1) to give substance to (2).

2. The 0-cell, 0-dimensional manifold (or point-pair), 1-cell, 1-dimensional manifold (or curve or circle) and the 1-dimensional complex, C_1, or linear graph.

3. Separation and sense in a 1-cell and a circle. Continuous transformation and deformation. Theorems on point sets, iteration, etc.

4. The matrix $\| \eta_{ij} \|$ of a C_1. The equations (modulo 2) determined by this matrix and the interpretation of their solutions as circuits in C_1.

5. If C_1 is connected there exists a linearly independent set of $\alpha_1 - \alpha_0 + 1$ simple circuits on which all others are linearly dependent. If $\alpha_1 - \alpha_0 + 1 = 0$, the graph is a tree.

6. Introduction of sense-relations, the matrix $\| \epsilon_{ij} \|$, and the corresponding equations. The number of solutions in a complete linearly independent set is again $\alpha_1 - \alpha_0 + 1$.

7. Notice of special investigations of linear graphs.

LECTURE II. THE n-DIMENSIONAL COMPLEX.

8. The n-cell, n-dimensional sphere, n-dimensional complex C_n. Closed and open or bounded complexes, n-dimensional circuits, n-dimensional manifolds M_n.

9. The matrices $\| \eta^1_{jk} \| = \mathrm{H}_1, \cdots, \| \eta^n_{jk} \| = \mathrm{H}_n$ of a C_n.

The sets of equations (modulo 2) determined by $H_1, \cdots, H_n$ and the interpretation of their solutions as circuits. $H_i \cdot H_{i+1} = 0$ $(i = 1, \cdots, n-1)$.

10. Definition of the numbers $R_0, R_1, \cdots, R_n$.

$$\alpha_0 - \alpha_1 + \cdots + (-1)^n \alpha_n = 1 + (-1)^n + (R_0 - 1) + \cdots + (-1)^n \cdot (R_n - 1).$$

If C_n determines a manifold,

$$\sum_{i=0}^{n} (-1)^i \alpha_i = 1 + (-1)^n + \sum_{i=1}^{n-1} (-1)^i (R_i - 1).$$

11. Sense relations and the Poincaré matrices $\| \epsilon_{jk}^1 \| = E_1, \cdots, \| \epsilon_{jk}^n \| = E_n$. Discrimination between one and two-sided complexes. The sets of linear equations corresponding to $E_1, \cdots, E_n$. Congruences and homologies. $E_i \cdot E_{i+1} = 0$ $(i = 1, 2, \cdots, n-1)$.

12. The Betti numbers $P_0, P_1, \cdots, P_n$ as defined by Poincaré. For any complex,

$$\sum_{i=0}^{n} (-1)^i \alpha_i = 1 + (-1)^n + \sum_{i=0}^{n} (-1)^i (P_i - 1).$$

For a two-sided manifold,

$$\sum_{i=0}^{n} (-1)^i \alpha_i = 1 + (-1)^n + \sum_{i=1}^{n-1} (-1)^i (P_i - 1).$$

For a one-sided manifold,

$$\sum_{i=0}^{n} (-1)^i \alpha_i = 1 + \sum_{i=1}^{n-1} (-1)^i (P_i - 1).$$

13. Theorems on matrices whose elements are integers. Reduction of $E_1, \cdots, E_n$ to normal form. Poincaré's coefficients of torsion. Relation between $R_0, \cdots, R_n, P_0, \cdots, P_n$, and the coefficients of torsion.

Lecture III. Manifolds of n Dimensions.

14. Singular complexes in a manifold M_n. Every closed C_k in an n-cell bounds a C_{k+1}.

15. The numbers R_i and P_i and the coefficients of torsion are the same for all complexes C_n into which an M_n may be subdivided, and hence are the same for all manifolds with

which M_n is in $(1-1)$ continuous reciprocal correspondence. Alexander's proof.

16. Dual complexes. Duality relation satisfied by the constants R_i for all manifolds and by P_i and the coefficients of torsion for two-sided manifolds. The characteristic, $\sum_{i=0}^{n}(-1)^i\alpha_i$, is zero if n is odd.

17. One-to-one correspondence between manifolds. Homeomorphism, homoömorphism, and internal transformation. Application of the theorems of separation of an n-cell.

18. Combinatorial proof of the theorems of separation of an n-cell by polyhedra. The theorem of Jordan and its generalizations.

Lecture IV. Two-Dimensional Manifolds.

19. Reductions of the complex defining a two-dimensional manifold. Normal forms and classification of surfaces.

20. The group of a two-dimensional manifold. Representation in the parabolic and hyperbolic planes.

21. Transformations and equivalence of curves on a surface. Homotopy and isotopy.

22. Continuous transformations and deformations. Theorems of Tietze.

Lecture V. The Group of a Complex.

23. The group defined by any complex or manifold.

24. Generating operations and essential relations. Theorem of Tietze.

25. The Cayley color diagram as developed by Dehn.

26. The group of a knot according to Dehn.

27. Relation to the older theories of knots.

Lecture VI. Three-Dimensional Manifolds.

28. Reductions of the complex defining a three-dimensional manifold.

29. A three-dimensional manifold as one cell with a singular boundary.

30. The Heegard diagram.

31. Any manifold may be decomposed into four cells without singularities.

32. The problem of classifying three-dimensional manifolds.
33. Particular three-dimensional manifolds. Riemann spaces. Poincaré spaces.

VIRGIL SNYDER.

NOTE ON THE ORDER OF CONTINUITY OF FUNCTIONS OF LINES.

BY DR. CHARLES ALBERT FISCHER.

(Read before the American Mathematical Society, September 4, 1916.)

IT has been proved that if a linear function of a line has continuity of the zeroth order, it can be expressed as the limit of a sequence of definite integrals,* as an integral of Stieltjes,† or as a Lebesgue integral.‡ As has been remarked by Bliss,§ many of the functions occurring in the calculus of variations do not have such continuity. The object of the present note is to show that if a function $U[y(x)]$ is linear and has continuity of the nth order, and if $y(x)$ is of class $C^{(n)}$, then $U[y(x)]$ is equal to the sum of a linear function of $d^n y(x)/dx^n$ which has continuity of the zeroth order, and a function of the values of $y(x)$ and its derivatives at an end point of the curve considered.

The proof is very simple. If $y(x)$ is of class $C^{(n)}$ it can be expressed as

$$y(x) = \int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} y^{(n)}(x_n)dx_n + \sum_{i=1}^{n} \frac{y^{(n-i)}(a)}{(n-i)!}(x-a)^{n-i}.$$

Then, since $U[y(x)]$, is linear,

$$U\left[\overset{b}{\underset{a}{y(x)}}\right] = V\left[\overset{b}{\underset{a}{y^{(n)}(x)}}\right] + \sum_{i=1}^{n} \frac{y^{(n-i)}(a)}{(n-i)!} U\left[\overset{b}{\underset{a}{(x-a)^{n-i}}}\right],$$

where the function

$$V[y^{(n)}(x)] = U\left[\int_a^x dx_1 \cdots \int_a^{x_{n-1}} y^{(n)}(x_n)dx_n\right]$$

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* Hadamard, "Leçons sur le Calcul des Variations," p. 299.
† Riesz, *Annales Scientifiques de L'Ecole Normale Supérieure*, vol. 28 (1911), p. 43.
‡ Fréchet, *Transactions Amer. Math. Society*, vol. 15 (1914), p. 140.
§ Bliss, *Proc. Nat. Acad. Sciences*, vol. 1 (1915), p. 173.

is seen to be linear and continuous of the zeroth order, and $U[(x-a)^{n-i}]$ is a constant.

To illustrate this, the function $U[y(x)]$ will be taken as the first variation of the integral

$$J = \int_a^b f(x, y, y')dx,$$

when $y(x)$ is replaced by $y(x) + \epsilon\eta(x)$. That is,

$$U[\eta] = \epsilon \int_a^b (f_y\eta + f_{y'}\eta')dx.$$

If $f(x, y, y')$ and $y(x)$ are of class $C^{(2)}$, and $\eta(a) = \eta(b) = 0$, this may be written

$$U[\eta] = \epsilon \int_a^b \left(f_y - \frac{d}{dx}f_{y'}\right)\eta dx.$$

If $f(x, y, y')$ and $y(x)$ are only of class C', this equation is not valid. In this case

$$U[\eta] = V[\eta'] = \epsilon \int_a^b \left(f_{y'} - \int_a^x f_y dx\right)\eta' dx,$$

which is linear and continuous of the zeroth order. It is evident that the expression

$$f_y - \frac{d}{dx}f_{y'}$$

is the Volterra derivative of the integral J considered as a function of $y(x)$, and it is approached with the second order.* Similarly

$$f_{y'} - \int_a^x f_y dx$$

is the derivative of J considered as a function of $y'(x)$. It is approached with the zeroth order.

This can easily be extended to functions of surfaces. For convenience it will be supposed that the function $U[z(x, y)]$ is linear and continuous of the second order, and that $z(x, y)$ is of class $C^{(2)}$ and defined over the region $(0 \leqq x \leqq 1;$ $0 \leqq y \leqq 1)$. It follows immediately that

* For a definition of the order of approach see Fischer, *Amer. Jour. of Mathematics*, vol. 35, no. 4 (1913), p. 383.

$$z(x, y) = \int_0^x dx_1 \int_0^y z_{xy}(x_1, y_1)dy_1 + \int_0^y dy_1 \int_0^{y_1} z_{yy}(0, y_2)dy_2$$
$$+ \int_0^x dx_1 \int_0^{x_1} z_{xx}(x_2, 0)\,dx_2 + z_y(0, 0)y + z_x(0, 0)x + z(0, 0).$$

Proceeding as before,

$$U[z(x, y)] = V[z_{xy}(x, y)] + V_1[z_{yy}(0, y)] + V_2[z_{xx}(x, 0)]$$
$$+ z_y(0, 0)U[y] + z_x(0, 0)U[x] + z(0, 0)U[1],$$

where V, V_1 and V_2 are linear and continuous of the zeroth order and $U[y]$, $U[x]$ and $U[1]$ are constants.

COLUMBIA UNIVERSITY,
June, 1916.

THE EQUATION OF A PLANE RATIONAL CURVE DEFINED BY PARAMETRIC EQUATIONS.

BY HERBERT WILLIAM RICHMOND.

THE explicit equation $F(x, y, z) = 0$ of a rational plane curve, defined by the parametric equations

$$x : y : z = A(t) : B(t) : C(t)$$

where $A(t)$, $B(t)$, $C(t)$ are polynomials of order n in t, is expressed by Salmon (Higher Plane Curves, § 44) as a determinant of $3n$ rows. In the April number of the BULLETIN Professor J. E. Rowe exhibits this equation as a symmetrical determinant of n rows, in which each element is a linear function of x, y, z. It has been my custom when lecturing upon algebraic geometry to obtain this form of equation as follows:

Let (x, y, z) be the coordinates of the point whose parameter is t, and let s be any value of the parameter; then

$$\begin{vmatrix} x & A(t) & A(s) \\ z & C(t) & C(s) \\ y & B(t) & B(s) \end{vmatrix} \quad (= \Delta \text{ say})$$

vanishes for every value of s. Imagine the determinant expanded and the factor $s - t$ removed. We have now a

polynomial in s of order $n - 1$, which vanishes for all values of s; the n coefficients of powers of s severally vanish. But each coefficient is a polynomial in t of order $n - 1$, every power of t being multiplied by a linear function of x, y, z. These powers of t may be eliminated by a determinant of n rows, which is the equation of the curve.

The linear function of (x, y, z) which in the rth row and sth column is the coefficient of $s^{n-r}t^{n-s}$ in the expanded form of $\Delta/(s - t)$, and is readily written down if need be. Just as Salmon's method is an adaptation of Euler's (or Sylvester's) method of eliminating a variable from two polynomials, so the foregoing is an adaptation of Cayley's statement of Bezout's method.

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SCIENTIFIC METHOD IN PHILOSOPHY.

Our Knowledge of the External World as a Field for Scientific Method in Philosophy. By BERTRAND RUSSELL, M.A., F.R.S., Late Fellow of Trinity College, Cambridge. Chicago, London, The Open Court Publishing Company, 1914. vii + 245 pp.

THE philosophic method in science is a thing familiar enough, in sound if not in fact. But what, pray, can be meant by "scientific method in philosophy"? That is what Mr. Russell essays to make clear, in outline at all events, in this book, and he does it partly by a general critique on the methodologies of philosophy and science and partly, indeed mainly, by handling certain specific problems of philosophy in the spirit and the manner of science. The chief ones among these illustrative examples belong to the general problem of the relation between the data of sense and the time, space, and matter of mathematical physics. The discussion is guided by a highly important and exceedingly difficult aim. The aim is to sketch a method available in all departments of philosophy and "adequate to yield whatever objective scientific knowledge it is possible to obtain." Traditional philosophy, however worthy of studious attention, has been too impatient; it has been too ambitious; its pretensions and claims have been grossly extravagant; it has not

been sufficiently disinterested; it has been too subjective, too personal, colored and often vitiated by individual tastes and temperaments. Such defects, which keep the great philosophic systems of the past from being scientific, are to be cured or avoided by means of a right philosophic, or scientific, method. What is this method? It is the method of logical analysis—a method first completely exemplified by Frege, a method copiously illustrated in the first three volumes of Whitehead and Russell's Principia Mathematica and, so far as the great conception of the world of physics as a construction rather than an inference is concerned, to be elaborately and precisely presented by Whitehead in the fourth volume of that work.

Mr. Russell's book, which is composed of the eight Lowell Lectures delivered by the author in Boston in 1914, is a rough, general, semi-popular sketch of the method in question. The presentation does not pretend to be accurate in all its details but it is confident of being sound in general and it is animated by the conviction that the method outlined "has great rewards to offer—triumphs as noteworthy as those of Newton and Darwin—and as important in the long run for the moulding of our mental habits."

The initial lecture, which is entitled "Current tendencies," is devoted to a clearing of the ground. The current tendencies dealt with are three: the classical tradition, evolutionism, and what Mr. Russell, for want of a better name, calls logical atomism. These are in conflict. The third is born of the modern critical movement in mathematics and is the type of philosophy that the author advocates. The first and the second types contend with one another and they are both of them antagonists of the third. The classical tradition seeks "to adapt to present needs the methods and results of the great constructive philosophers from Plato downwards," thus owning such names as Kant and Hegel and represented in our own day by the Appearance and Reality of Mr. Bradley. It is doomed by its too great reliance on pure reason and the meagreness of its imagination in conceiving the universe. The universe as revealed by modern science it does not know. It is dying at the hands of discovered natural facts. Mr. Russell would not, I suppose, deny that this philosophy has performed an immense service in testing the resources of reason. The experiment had to be made, for there was no

way of knowing a priori that reason, which had succeeded in building an immortal geometry, was not competent to construct a true philosophy of the entire world.

Evolutionism, the regnant philosophy of our time, is a reaction against the classical tradition. It springs from and allies itself with natural knowledge, as in the works of Herbert Spencer and Bergson. But it "is not a truly scientific philosophy, either in its methods or in the problems which it considers." It is not sufficiently disinterested; it is too easy-going; it does not know the severity of logical discipline. To understand the world it is necessary to understand change and continuity. But biology, physics, chemistry, and the like can not enlighten us here. The problem is foreign to natural science. Again, evolutionism is primarily interested in human destiny, but scientific philosophy is not. It is more detached. It is not Baconian. It has no essential or primary concern with questions that the other sciences may answer. It is not tethered to mundane interests. Incidentally it may help, as it may be helped by, the other sciences, but it appeals to none except such as desire, beyond all else, "to escape from intellectual bewilderment." Towards mysticism its attitude is this: It does not deny the genuineness of the mystic's alleged insights and intuitions of truth beyond the world of sense, but when the mystic denies truth to the deliverances of sense and normal perception it submits a challenge, it asks for evidence. And unlike Bergsonianism, it regards the logical analysis practiced by a Cantor or a Frege as superior to the instinct of a wasp or of a duck that will lead a brood of chicklets to a pond.

The second lecture bears the somewhat challenging title, "Logic as the essence of philosophy." In what sense is the implied thesis to be understood or to be justified? Certainly not at all if logic is to be identified with Aristotelian syllogistic. Mr. Russell does not pause to give due credit to the great logical work of Aristotle, possibly because he thinks the world has already done more than justice in that matter. But a great outburst of genius often blinds centuries of men and certainly Aristotle fettered the mediaeval mind. In a measure the fetters were broken by Bacon and Galileo. Yet Bacon's conception of the nature and proper function of induction was superficial and mistaken. Induction can not replace deduction, but can widen its scope. Deduction is not

all of it syllogistic. But how deduce by induction? What principle justifies inference, for example, from past to future sunrises? If, with Mill, we say the law of causation, then we must ask what justifies belief in that law. That the law is known a priori is rendered improbable by the formulation of the law, which runs somewhat as follows: "A causal law is any general proposition in virtue of which it is possible to infer the existence of one thing or event from the existence of another or of a number of others." If we suppose causal law to be an assumption, then evidently it can not justify inference in any fundamental or ultimate way. Is it an empirical generalization? Then, how justify such generalization? Not by enumeration, for this can at best yield probability—whatever that is—, nor by any other empirical means, for generalization transcends experience. Inference from the observed to the unobserved, if valid, must rest, not on any law of causality, but upon the principle of induction. What is this principle? It is hard to state it quite satisfactorily but it is much like this: "If, in a great number of instances, a thing of a certain kind is associated in a certain way with a thing of a certain other kind, it is probable that a thing of the one kind is always similarly associated with a thing of the other kind; and as the number of instances increases, the probability approaches indefinitely near to certainty." The proposition may be false; it admits of neither proof nor disproof. Yet it is the only justification that the kind of inference in question admits of.

But it is not to be admitted, because general propositions can not be empirically established, that we can not know any general propositions to be true. We have such knowledge and it belongs to logic. Logic consists of two parts, and we are now to see wherein and why it is the essence of philosophy. "The first part investigates what propositions are and what forms they may have." It "enumerates the different kinds of atomic propositions," that is, propositions which, when asserted, assert that a certain thing has a certain quality or that certain things have a certain relation; it enumerates the different kinds of "molecular" propositions, or propositions containing "conjunctions—*if*, *or*, *and*, *unless*, etc."; and it enumerates the various kinds of general propositions, such as, all equilateral triangles are equiangular; and so on. For logic the content of propositions is of no importance, but form is

all-important. The second part of logic "consists of certain supremely general propositions which assert the truth of all propositions of certain forms." An example of such supreme propositions is: "If anything has a certain property, and whatever has this property has a certain other property, then the thing in question has the other property." Such a proposition is true, and is known to be true, but can not be established empirically. This second part of logic is the first part of mathematics; it is the first part of logic that makes logic the essence of scientific philosophy. One is reminded of the saying of Leibnitz: "Ma métaphysique est toute mathématique."

It is in the third lecture, "On our knowledge of the external world," that Mr. Russell comes to close grips with his task and best indicates the method of "logical analysis" or the philosophy of "logical atomism" by operating it on certain fundamental problems. From the earliest times philosophy has from various motives found it necessary to discredit sense-data and to suppose something more real behind the veil. And modern science and especially physics (taken in a comprehensive sense) have not escaped the like necessity. Atoms, molecules, electrons, points, instants, time, space (the time and space of mathematical physics)—none of these things and their kind is known or knowable to sense or perception. Neither can they be logically inferred from the data of sense. They can not be proved to exist nor can their non-existence be shown logically. Why assume them? It is not necessary to do so, and, if not necessary, such assumption is forbidden by the principle of Occam's Razor. What, then, is proposed instead? Instead of illegitimately pretending to *infer* such things as the points, instants, matter, time, and space of mathematical physics and instead of *assuming* the existence of such things, it is proposed and shown to be possible to *construct* them from the data furnished by sense. That done, we shall know them to exist in whatever sense we may suppose constructs to exist, because we have made them, and physics will cease to be an illegitimate inference or a structure based on pure assumption and will be instead a structure built of and upon sense data, the only facts immediately known or knowable respecting the external world. How is the task of construction to be performed? A hint, by means of one example, must here suffice.

Let us confine ourselves to the sense of sight and see how we may construct a corresponding space. The sketch, for the want of room, must needs be rough. Each of many minds, looking out from its own view-point, sees at each moment a three-dimensional world. No two minds behold precisely the same thing. Each has its own "perspective," its own "private world." A mind at an intermediate point of view will have an intermediate private world. Such perspectives or worlds may be correlated by the principle of similarity, for whilst no two are identical, they may resemble each other in various degrees. Two that are much alike may be said to be "near" one another in space but this space is entirely different from "the spaces inside the two perspectives." It is a relation, a relation between the two private worlds or spaces. Between these we can in thought intercallate others more and more similar and hence nearer and nearer together. In this way public space, which consists of relations between private spaces, can be made continuous. Thus space is a constructed affair: it is a system, a system of perceived or perceivable private worlds. In ways quite analogous, we can construct a "thing," a "point," "here," "instant," "time" and so on and on. The delineation of the process with critical remarks as to its philosophic significance occupies the fourth lecture, "The world of physics and the world of sense."

The fifth lecture is devoted to "The theory of continuity." The author ranges himself on the side of those who regard sensation as a continuous function of stimulus though he does not think the doctrine capable of being established empirically. As this doctrine involves the necessity of supposing that two sensations may be different despite the fact that their subject can not discriminate them, it appears to involve a contradiction in terms or a verbal quibble or a verbal confusion. It seems sound to say that two sensations that are sensed as the same are the same, *as* sensations.

Then follow two lectures devoted to the notion of infinity—"the problem of infinity considered historically" and "the positive theory of infinity." Though these lectures are intensely interesting, they contain but little, except some novelty of exposition, that is likely to be new to the student acquainted with the literature of Cantorism. This is not to imply that they have not very great value as exhibiting the nature and power of logical atomism as a method and type of philosophy.

The work closes with a lecture treating "The notion of cause, with applications to the free-will problem." The discussion is very acute, the finest in the book. Hume's classical analysis is resumed and shown to require extension and this is undertaken. The vulgar notion of a cause as compelling its effect must be abandoned as having no warrant in logic and no essential rôle in natural science. A cause may as well accompany or follow its effect as precede it. Far more important than the notion of cause is that of causal law, which has been above stated in one form. It is acutely contended that, as causes do not compel, the acts of will may be caused without being externally coerced, and that omiscience, including knowledge of the entire future, is consistent with every thing in freedom that is worth preserving.

The book as a whole must be judged as an important contribution to the science of philosophy even if the reader must remain convinced that much that is destined to continue to be called philosophy will not, through logical analysis or other means, yield solid, scientific, objective knowledge. Personal idiosyncrasies are themselves facts and they are often more interesting than, and quite as important as, generic results that ignore them. In the future, as in the past, the value of philosophy will consist, not wholly in propositions established by it, but largely in philosophizing. Let the two kinds flourish side by side, but let them not be confounded.

CASSIUS J. KEYSER.

SHORTER NOTICES.

Combinatory Analysis. By Major PERCY A. MACMAHON. Volume 1. Cambridge University Press, 1915. 300 + xix pp.

THE author states that "the object of this work is, in the main, to present to mathematicians an account of theorems in combinatory analysis which are of a perfectly general character, and to show the connection between them by as far as possible bringing them together as parts of a general doctrine. Little attempt has been hitherto made either to make a general attack upon the territory to be won or to coordinate and arrange the ground that has been already gained. The combinatory analysis as considered in this work occupies the ground between algebra, properly so

called, and the higher arithmetic. The methods employed are distinctly algebraical and not arithmetical."

Section I is devoted to the study of symmetric functions. The theory is developed with particular reference to the theory of the partitions of numbers. In general we may have in view the symmetric function

$$\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\cdots\alpha_s^{p_s},$$

the whole function consisting of a number of terms similar to the one attached to the sign of summation; in each term we may take $p_1, p_2, \cdots, p_s$ to be in descending order of magnitude, but the quantities α, therein appearing, will be any s selected from $\alpha_1, \alpha_2, \cdots, \alpha_n$ in any permutation. It thus appears that the succession of numbers $p_1, p_2, \cdots, p_s$ in descending order of magnitude is a sufficient specification of the symmetric function and, if

$$p_1 + p_2 + \cdots + p_s = w,$$

we may regard the function as denoted by the partition $(p_1p_2\cdots p_s)$ of the number w. It is now shown that the general theory of combinations is essentially involved in the algebra of monomial symmetric functions. Every multiplication of monomial symmetric functions involves a theorem in combinations.

Further developments of the theory of symmetric functions are made by introducing the two differential operators of Hammond:

$$d_\lambda = \frac{d}{da_\lambda} + a_1\frac{d}{da_{\lambda+1}} + a_2\frac{d}{da_{\lambda+2}} + \cdots$$

$$D_s = \frac{1}{s!}(d_1^s),$$

where (d_1^s) indicates a symbolic multiplication as usual in Taylor's expansion. In Chapters IV and V it is shown that the algebra of these operators is parallel with the algebra of symmetric functions.

Section II is a generalization of the theory of Section I. The theory of symmetric functions is associated not with a number but with the partition of a number. The theory of Section I is that particular part of a more general theory which is associated with that partition of a number which is

composed wholly of units. Hammond's operators d and D are given an extended field of operation and, in the enlarged theory, are shown to be important instruments for multiplying and evaluating the symmetric functions that present themselves. In Chapter IV binomial coefficients are treated as symmetric functions denoted by partitions with zero parts, and the formation of symmetrical tables involving them is explained.

In Section III the enumeration of combinations and permutations is treated from the point of view supplied by the theory of symmetric functions. In Chapter II a theorem is established termed "a Master Theorem from the masterly and rapid fashion in which it deals with various questions otherwise troublesome to solve. Many illustrations of its power are given. In particular in Chapter III it is shown to supply instantly the solution of the generalized 'problème des rencontres.' In finding expressions for the sum of powers of binomial coefficients it is singularly effective."

Chapter V is devoted to lattice permutations. Consider an assemblage of letters $\alpha^p\beta^q\gamma^r \cdots$ in which the numbers $p, q, r, \cdots$ are in descending order of magnitude. This particular permutation of the assemblage can be denoted by a regular graph consisting of rows of dots. The successive rows will have $p, q, r, \cdots$ dots respectively and the graph is the same as serves to denote the partition $(pqr \cdots)$ of the number $p + q + r + \cdots$. Such a graph may be

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for $p = 6, q = 4, r = 1$.

If we take *any* permutation of $\alpha^p\beta^q\gamma^r \cdots$ we shall arrive finally at the same graph by proceeding from left to right of the permutation and placing a dot in the first row, or in the second, or in the third according as we reach a letter α, β, or γ, etc.

The author states that lattice permutations "are in themselves interesting but are of importance principally because they are shown in Volume II to supply the key to solutions of certain questions which involve arrangements, under given conditions, of numbers in spaces of two and three dimensions; numbers not arranged at points in a straight line but at the

crossing points of lines which compose lattices in two and three dimensions."

Section IV is devoted to the theory of the compositions of numbers. The word connotes a partition in which account is taken of the order of occurrence of the parts. Thus the partition a, b, c of the number $a + b + c$ would have six compositions $abc, acb, bac, bca, cab, cba$ involving the parts a, b, c. Chapter I treats with unipartite numbers; Chapter II with multipartite numbers, a graphical representation being given; Chapter III with the graphical representation of the compositions of tripartite and multipartite numbers. Chapter IV gives a complete solution of Simon Newcomb's problem: A pack of cards of any specification is taken—say there are p cards marked 1, q cards 2, r cards 3, and so on—and being shuffled is dealt out on a table; so long as the cards that appear have numbers that are in ascending order of magnitude, equality of numbers counting as ascending order, they are placed together in one pack, but directly the ascending order is broken a fresh pack is commenced and so on until all the cards have been dealt. The probability that there will result exactly m packs or at most m packs is required. Chapter V gives a generalization of the foregoing theory.

In Section V the subject of the perfect partitions of numbers is dealt with as a necessary preliminary to the discussion of arrangements upon a chess board. The latter, as involving conditions to be satisfied by the numbers appearing in a row or in a column, are of the magic square nature. Operators and functions are designed with the object of discovering the laws of formation of the systems of diagrams to which they naturally lead. As the simplest example choose the operation d/dx and the function x^n. We observe that if x^n be written out in the form $xxx \cdots x$ the operation is equivalent to writing unity for one of the x's in all possible ways and adding the results together. We obtain nx^{n-1} because one x can be selected in n different ways. So operating n times successively with d/dx we dissect the operation into $n!$ distinct operations and these give rise to $n!$ distinct diagrams. For consider a square of n^2 compartments; we may place a unit at the intersection of the rth row and cth column to indicate that when d/dx was operating for the rth time our process was to substitute unity for the cth x counting from the left. So we obtain $n!$ diagrams which possess the property that n units

appear one in each row and one in each column. We thus by the operation upon the function enumerate the diagrams with this property.

The problem of the Latin square is solved. This problem is to place n different letters $a, b, c, \cdots$ in each row of a square of n^2 compartments in such wise that, one letter being in each compartment, each column involves the whole of the letters. The number of arrangements is required. "The question is famous because, from the time of Euler to that of Cayley inclusive, its solution was regarded as being beyond the powers of mathematical analysis. It is solved without difficulty by the method of differential operators of which we are speaking. In fact it is one of the simplest examples of the method which is shown to be capable of solving questions of a much more recondite character." Most of the operators are those of the infinitesimal calculus and the numbers involved in the partitions of the functions are positive integers, excluding zero. If we admit zero as a part in the partitions, we have to do with the operations of the calculus of finite differences.

In Section VI the theory of distributions is applied to the enumeration of the partitions of multipartite numbers.

The author states that "In the present volume there appears a certain amount of original matter which has not before been published. It involves the author's preliminary researches in combinatory theory which have been carried out during the last thirty years."

The book is not intended for class room work. There is an absence of problems, but plenty of exercise for one who carries out in detail all of the work indicated. There is no index but a full table of contents. There are errors. It is not our intention to catalogue these. We refer to one merely to support our statement: on page 5 last line for $a_2{}^p$ write $a_2{}^{p_2}$. At certain places conditions are implied which we think should be explicitly stated. We mention but one: page 4 line 3 we should have $\lambda_1 + 2\lambda_2 + 3\lambda_3 + \cdots + s\lambda_s = s$. Problems concerning symmetry of determinants find no place in the present treatise. There are thirteen pages of tables at the end exhibiting in a condensed form some of the results of the foregoing theory. In general the work is well written and clear.

W. V. Lovitt.

Plane Analytic Geometry, with introductory chapters on the Differential Calculus. By Professor MAXIME BÔCHER. Henry Holt and Company, New York, 1916. vii + 235 pp.

THIS compact and clearly written textbook gives the essential properties of the conic sections and ends with two chapters on the differentiation of algebraic functions with applications.

The lemniscate is the only curve of higher degree which is discussed at any length, but its graph is correct, thereby differing from the representations commonly published. Other curves are barely mentioned, nor has the book an index. Thus the student of engineering especially must wait for a later course to unfold cardioids, catenaries, spirals, and witches.

The preface states that "Analytic geometry, if properly taught, is a difficult subject, and concentration on a few of its important principles is necessary if mastery is the aim," and later, "It is only by degrading it to a course in graphics (curve plotting, numerical problems, etc.), that a course in analytic geometry can be made easy to the average student."

The student will have to be careful to note that in a right-handed system of coordinates, a positive (left-handed) rotation changes the OX axis into the OY axis. In a note the author states that "The word 'normal' (in Hesse's normal form of the straight line equation) is here used in the sense of 'standard.' It has nothing to do, as some American textbooks have implied, with the normal to a curve or line."

Throughout the book, curves with no real points are consistently referred to as "no locus," and definitions are so framed as to exclude degenerate cases. The chapter on determination of loci is short but explicit, while that on the general equation of the second degree gives a detailed and complete discussion, ending with the problem of a conic through five points.

For the student proceeding to the calculus, a course in solid analytic geometry must follow this text, but otherwise one who has finished this book will be excellently prepared.

F. H. SAFFORD.

NOTES.

THE twenty-third annual meeting of the American Mathematical Society will be held at Columbia University on Wednesday and Thursday, December 27–28, 1916, in connection with the meeting of the American association for the advancement of science. At this meeting President ERNEST W. BROWN will deliver his presidential address, the subject of which will be "The relation of mathematics to the natural sciences." Titles and abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary by December 13. Abstracts to be printed in advance of the meeting should be sent in by December 6.

A NEW edition of the List of Officers and Members of the Society is now in preparation and will be issued early in January. Blanks for furnishing necessary information have been sent to the members. A prompt response will contribute materially to the correctness and completeness of the List.

THE first summer meeting of the Mathematical Association of America was held under the presidency of Professor E. R. HEDRICK at the Massachusetts Institute of Technology, Cambridge, Mass., on Friday and Saturday, September 1–2, thus immediately preceding the summer meeting of the American Mathematical Society. The attendance numbered 126, including 110 members of the Association. The programme included the following papers: E. V. HUNTINGTON: "The teaching of elementary dynamics"; D. E. SMITH: "The history of mathematical recreations"; J. N. VAN DER VRIES: "Combined courses in mathematics for college freshmen"; F. S. WOODS: "Combined courses in mathematics for freshmen in technical schools." An announcement of plans was made by the committee on mathematical requirements, and a preliminary report was received from the bureau of information. The dinner on Friday evening was attended by about 80 members and friends. The Association will hold its next meeting at Columbia University on Friday and Saturday, December 29–30.

THE sixty-ninth meeting of the American association for

the advancement of science will be held in New York during the week December 26–30 under the presidency of Dr. C. R. VAN HISE. Professor L. P. EISENHART is vice-president, and Professor F. R. MOULTON secretary of Section A.

THE British association for the advancement of science held its annual meeting at Newcastle September 5–9, under the presidency of Sir ARTHUR EVANS. Professor A. N. WHITEHEAD was president of Section A: mathematical and physical science. Professor Whitehead's address before the section, on "The organisation of thought," recently appeared in *Science* (September 22).

THE closing (October) number of volume 38 of the *American Journal of Mathematics* contains the following papers: "Conditions for the complete reducibility of groups of linear substitutions," by HENRY TABER; "On sextic surfaces having a nodal curve of order 8," by C. H. SISAM; "A theorem connected with irrational numbers," by W. D. MACMILLAN; "On the representation of arbitrary functions by definite integrals," by W. B. FORD; "Metric properties of nets of plane curves," by H. R. KINGSTON; "On a special elliptic ruled surface of the ninth order," by H. C. GOSSARD.

ON the occasion of his seventieth birthday, December 7, 1915, Professor A. Voss, of the University of Munich, received from the Munich technical high school the honorary degree of doctor of technical sciences.

PROFESSOR M. PASCH, of the University of Giessen, celebrated the fiftieth anniversary of his doctorate on August 21, 1915. On this occasion the University of Breslau renewed his diploma.

THE birthplace of WEIERSTRASS in Osterfelde in Westphalia has recently been marked by a memorial tablet.

DEAN T. F. HOLGATE, of Northwestern University, has been elected ad interim president of the University.

PROFESSOR J. J. LUCK, of Vanderbilt University, has been appointed adjunct professor of mathematics in the University of Virginia.

MR. C. H. CLEVENGER has been appointed instructor in mechanics and mathematics in the school of mines of the University of Minnesota.

MR. C. N. REYNOLDS has been appointed instructor in mathematics at Wesleyan University.

DR. P. R. RIDER has been appointed instructor in mathematics in Washington University.

MR. J. J. TANZOLA, of Columbia University, has been appointed instructor in mathematics in the U. S. Naval Academy.

AT Dartmouth College, Dr. C. H. FORSYTH, of the University of Michigan, has been appointed instructor in mathematics. Dr. F. J. MCMACKIN has resigned his instructorship in mathematics to engage in high school work.

PROFESSOR F. PRYM, of the University of Würzburg, died December 15, 1915, at the age of seventy-four years.

PROFESSOR P. DUHEM, of the University of Bordeaux, died in September at the age of sixty-seven years.

THE death is announced of Dr. William ESSON, Savilian professor of mathematics in the University of Oxford, at the age of eighty-eight years.

PROFESSOR S. B. MACLAREN, of University College, Reading, died August 14, 1916, from wounds received in battle.

PROFESSOR L. L. CONANT, of the Worcester Polytechnic Institute, was killed in a street accident on October 11, 1916, at the age of fifty-nine years. Professor Conant had been a member of the American Mathematical Society since 1892.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

HALPHEN (G. H.). Œuvres, publiées par les soins de C. Jordan, H. Poincaré, E. Picard, avec la collaboration de E. Vessiot. Tome 1. Paris, Gauthier-Villars, 1916. 8vo. 44 + 570 pp. Fr. 20.00

JONES (W. H. S.). Scientific method in schools: a suggestion. Cambridge, University Press, 1916. 36 pp. 1s.

JORDAN (C.). See HALPHEN (G. H.).

LINEHAN (P. H.). Contributions to equilong geometry. (Diss., Columbia.) Lancaster, Pa., P. H. Linehan, 1916. 4to. 6 + 38 pp. Paper. $1.00

MACAULAY (F. S.). The algebraic theory of modular functions. Cambridge, University Press, 1916. 14 + 112 pp. 4s. 6d.

PICARD (E.). See HALPHEN (G. H.).

POINCARÉ (H.). See HALPHEN (G. H.).

SCHWALLER (R.). Etude sur les nombres. Paris, Librairie de l'Art indépendant, 1916. 12mo. Fr. 1.50

VESSIOT (E.). See HALPHEN (G. H.).

II. ELEMENTARY MATHEMATICS.

BARNARD (S.) and CHILD (J. M.). Elements of geometry. Parts 5–6. London, Macmillan, 1916. 2s.

BEY (M.). Principes nouveaux de calculs rapides sans l'emploi des logarithmes. Paris, Desforges, 1915. 8vo.

CHILD (J. M.). See BARNARD (S.).

HERVÉ (E. L.). Conversion des mesures étrangères en mesures françaises. Paris, Challamel, 1916. 8vo. Fr. 3.00

LEIGH (C. W.). See PALMER (C. I.).

LEMOINE (A.). Arithmétique. Cours supérieur et complémentaire, brevet élémentaire. Paris, Hachette, 1916. 18vo. Cartonné. Fr. 1.80

LENNES (N. J.). See SLAUGHT (H. E.).

MARSH (H. W.). Interpolated six-place tables of the logarithms of numbers and the natural and logarithmic trigonometric functions. New York, Wiley, 1916. 8vo. 14 + 155 pp. Cloth. $1.25

MILLIS (J. F.). See STONE (J. C.).

PAILHADE (J. DE R.). Sur les avantages et les progrès de la mesure décimale française des angles. Nice, 1916.

PALMER (C. I.) and LEIGH (C. W.). Plane and spherical trigonometry with 5-place tables. 2d edition. New York, McGraw-Hill, 1916. 8vo. 186 + 132 pp. $1.50. Text alone, $1.00. Tables alone, $0.75.

PATERSON (W. E.) and TAYLOR (E. O.). Elementary geometry, theoretical and practical, covering Stage III of the recommendations of the Board of Education Circular 711. Vol. 1, 1914: Triangles and quadri-

laterals; Vol. 2, 1915: Circle and similar figures. Now issued in one volume. Oxford, Clarendon Press, 1916. Cr. 8vo. 328 pp. 3s.

PHILIP (M.). Algebra through simple equations. New York, Hinds, Hayden and Eldredge, 1916. 6 + 52 pp. $0.48

ROGER-COURT (—.). Arithmétique. Cours moyen. Certificat d'études. Livre du maître. Paris, Colin, 1916. 12mo. 590 pp. Cartonné. Fr. 4.00

ROORDA VAN EYSINGA (H.). Cours de mathématiques élémentaires. III: Manuel de géométrie. Théorie et problèmes. Paris, Payot, 1916. 12mo. 194 pp. Fr. 3.00

ROSENBERG (F.). Preliminary geometry. Oxford, Clarendon Press, 1916. 6 + 220 pp. 2s.

SLAUGHT (H. E.) and LENNES (N. J.). Intermediate algebra. Boston, Allyn and Bacon, 1916. 8vo. 249 pp. $0.75

STONE (J. C.) and MILLIS (J. F.). Plane geometry. Chicago, B. H. Sanborn, 1916. 8vo. 10 + 278 pp. $0.85

TAYLOR (E. O.). See PATERSON (W. E.).

III. APPLIED MATHEMATICS.

BAUSCH AND LOMB OPTICAL COMPANY. Metro manual; a handbook for engineers, containing technical information regarding the construction, adjustment and use of transits, tachymeters, theodolites, alidades, levels, etc. (9th and revised edition of Saegmuller vest-pocket handbook.) Rochester, N. Y., Bausch and Lomb, 1915. 199 + 37 pp.

BROWN (J. C.). See FINNEY (H. A.).

BURROUGHS ADDING MACHINE COMPANY. Instructions for operating and using Burroughs adding and listing machines, class three hundred. 9th edition. Detroit, Mich., Burroughs, 1915. 4to. 31 pp.

CARLIER (J.). La résolution de l'équation de traction d'un véhicule en mouvement par le "puissance-mètre." Paris, Le Génie civil, 1915. 8vo.

CONNAISSANCE des temps ou des mouvements célestes à l'usage des astronomes et des navigateurs, pour l'an 1918, publiée par le Bureau des Longitudes. Paris, Gauthier-Villars, 1916. Cartonné. Fr. 4.75

DICK (F. J.). Ancient astronomy in Egypt and its significance. (Papers of the School of Antiquity, University Extension Series, No. 7.) Point Loma, Cal., Aryan Theosophical Press, 1916. 4to. 18 pp.

DUHEM (P.). Le système du monde. Histoire des doctrines cosmologiques de Platon à Copernic. Tome 4. Paris, Hermann, 1916. Royal 8vo. 4 + 597 pp. Fr. 19.00. Edition sur papier de Hollande. Fr. 25.00

FINNEY (H. A.) and BROWN (J. C.). Modern business arithmetic. New York, Holt, 1916. 8vo. 298 pp. $0.85

GRIALOU (J.). Cours d'hydraulique. Paris, Gauthier-Villars, 1916. 8vo. 6 + 455 pp. Fr. 20.00

KARAPETOFF (V.). Engineering applications of higher mathematics. New York, Wiley. 8vo. Part 1: Problems on machine design, 1912. 14 + 69 pp. Part 2: Problems on hydraulics, 1916. 16 + 103 pp. Part 3: Problems on thermodynamics, 1916. 16 + 113 pp. Part 4:

Problems on mechanics of materials, 1916. 4 + 81 pp. Part 5: Problems on electrical engineering, 1916. 7 + 65 pp.
Each part $0.75

MADSEN (V. H. O.). Le service géodésique en Danemark (1816–1916). Copenhagen, Bianco-Luno, 1916. 4to.

MONCETON (J. H.). Practical geometry. 4th edition. New York, W. T. Comstock, 1915. 8vo. 13 + 82 pp. $1.00

PALMER (A. R.) and STEPHENSON (J.). Commercial arithmetic and accounts. London, Bell, 1916. Part 1: 14 + 292 + 56 pp. Part 2: 11 + 293–514 + 57–154 pp.

RELATORIO ASTRONOMICO. Uma das maiores conquistas de todos os tempos. Lisboa, A. de Mandonça, 1916. 8vo.

SAEGMULLER. See BAUSCH.

SMITH (W. G.). Practical descriptive geometry. 2d edition, enlarged and reset. New York, McGraw-Hill, 1916. 8vo. 10 + 256 pp. $2.00

STEPHENSON (J.). See PALMER (A. R.).

TRIMBLE (C. J. A.). See USHERWOOD (T. S.).

USHERWOOD (T. S.) and TRIMBLE (C. J. A.). Practical mathematics for technical students. Part 2. London, Macmillan, 1916. Cr. 8vo. 7s. 6d.